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A NOTE ON TYPE 2 DEGENERATE MULTI-POLY-BERNOULLI POLYNOMIALS OF THE SECOND KIND

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Abstract. In this paper, we introduce type 2 degenerate multi-poly-Bernoulli polynomials of the second kind which are defined by using the degenerate multi polyexponential function. We investigate some properties of those numbers and polynomials. Also, we give some identities and relations for the degenerate multi-poly-Bernoulli polynomials and numbers of the second.

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1. Introduction

The polyexponential function, as an inverse to the polylogarithm function [13] is defined by Kim-Kim to be

$$\mathrm{Ei}_k(x)=\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!n^k},(k\in\mathbb{Z}). \tag{1.1}$$

For $k=1$, (1.1) gives

$$\mathrm{Ei}_1(x)=\sum_{n=1}^{\infty}\frac{x^n}{n!}=e^x-1. \tag{1.2}$$

For any nonzero $\lambda\in\mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by

$$e_\lambda^x(t)=\sum_{n=0}^{\infty}(x)_{n,\lambda}\frac{t^n}{n!},(\text{see, [3, 4, 5, 6]}), \tag{1.3}$$

where $(x)_{0,\lambda}=1$, $(x)_{n,\lambda}=x(x-\lambda)\cdots(x-(n-1)\lambda),(n\geq 1)$.

Note that

$$\lim_{\lambda\to 0}e_\lambda^x(t)=\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}=e^{xt}.$$

In [1, 2], Carlitz introduced the degenerate Bernoulli polynomials given by

$$\frac{t}{e_\lambda(t)-1}e_\lambda^x(t)=\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\beta_n(x;\lambda,)\frac{t^n}{n!}. \tag{1.4}$$

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When $x=0$, $\beta_n(\lambda)=\beta_n(0;\lambda)$ are called the degenerate Bernoulli numbers.

Note that

$$\lim_{\lambda\longrightarrow 0}\beta_n(x;\lambda)=B_n(x).$$

Kim *et al.* [17] introduced the modified degenerate polyexponential function given by

$$\mathrm{Ei}_{k,\lambda}(x)=\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}}{(n-1)!n^k}x^n. \tag{1.5}$$

Note that

$$\mathrm{Ei}_{1,\lambda}(x)=\sum_{n=1}^{\infty}\frac{(1)_{n,\lambda}}{n!}x^n=e_\lambda(x)-1.$$

The type 2 degenerate poly-Bernoulli polynomials [17] are defined by means of the following generating function

$$\frac{\mathrm{Ei}_{k,\lambda}\left(\log_\lambda(1+t)\right)}{e_\lambda(t)-1}e_\lambda^x(t)=\sum_{n=0}^{\infty}B_{n,\lambda}^{(k)}(x)\frac{t^n}{n!},(k\in\mathbb{Z}). \tag{1.6}$$

When $x=0$, $B_{n,\lambda}^{(k)}=B_{n,\lambda}^{(k)}(0)$ are called the type 2 degenerate poly-Bernoulli numbers.

In [15], the degenerate Bernoulli polynomials of the second kind are defined by

$$\frac{t}{\log_\lambda(1+t)}(1+t)^x=\sum_{n=0}^{\infty}b_{n,\lambda}(x)\frac{t^n}{n!},\text{(see [27])}. \tag{1.7}$$

Note that $\lim_{\lambda\to 0}b_{n,\lambda}(x)=b_n(x),(n\geq 0)$.

For $k_1,k_2,\cdots k_r\in\mathbb{Z}$, the multiple polylogarithm [26] is defined by

$$\mathrm{Li}_{k_1,k_2,\cdots,k_r}(x)=\sum_{0<n_1<n_2<\cdots<n_r}\frac{x^{n_r}}{n_1^{k_1}n_2^{k_2}\cdots n_r^{k_r}}, \tag{1.8}$$

where the sum is over all integers $n_1,n_2,\cdots,n_r$ satisfying $0<n_1<n_2<\cdots<n_r$.

The multi-poly-Bernoulli polynomials of the second kind are defined by

$$\frac{r!\mathrm{Li}_{k_1,k_2,\cdots,k_r}(1-e^{-t})}{[\log(1+t)]^r}(1+t)^x=\sum_{n=0}^{\infty}b_n^{(k_1,k_2,\cdots k_r)}(x)\frac{t^n}{n!},\text{(see [10, 11, 26])}. \tag{1.9}$$

When $x=0$, $b_n^{(k_1,k_2,\cdots k_r)}=b_n^{(k_1,k_2,\cdots k_r)}(0)$ are called the multi-poly-Bernoulli numbers of the second kind.

The degenerate multi-poly-Bernoulli polynomials given by

$$\frac{r!\mathrm{Li}_{k_1,k_2,\cdots,k_r}(1-e^{-t})}{(e_\lambda(t)-1)^r}e_\lambda^x(t)=\sum_{n=0}^{\infty}\beta_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)\frac{t^n}{n!}, \text{(see [23])}. \tag{1.10}$$

When $x=0$, $\beta_{n,\lambda}^{(k_1,k_2,\cdots k_r)}=\beta_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(0)$ are called the degenerate multi-poly-Bernoulli numbers.

For $k_1,k_2,\cdots,k_r\in\mathbb{Z}$, the degenerate multiple polyexponential function is defined by (see [24])

$$\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}(x)=\sum_{0<n_1<n_2<\cdots<n_r}\frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}x^{n_r}}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}}, \tag{1.11}$$

where the sum is over all integers $n_1,n_2,\cdots,n_r$ satisfying $0<n_1<n_2<\cdots<n_r$.

Kim *et al.* [24] introduced and studied the degenerate multi-poly-Genocchi polynomials are defined by

$$\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}(\log_\lambda(1+t))}{(e_\lambda(t)+1)^r}e_\lambda^x(t)=\sum_{n=0}^{\infty}g_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)\frac{t^n}{n!}. \tag{1.12}$$

When $x=0$, $g_{n,\lambda}^{(k_1,k_2,\cdots k_r)}=g_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(0)$ are called the degenerate multi-poly-Genocchi numbers.

The degenerate Stirling numbers of the first kind (see [14]) are defined by

$$\frac{1}{k!}(\log_\lambda(1+t))^k=\sum_{n=k}^{\infty}S_{1,\lambda}(n,k)\frac{t^n}{n!},\ (k\geq 0). \tag{1.13}$$

It is notice that $\lim_{\lambda\to 0}S_{1,\lambda}(n,k)=S_1(n,k)$, where $S_1(n,k)$ are called the Stirling numbers of the first kind given by

$$\frac{1}{k!}(\log(1+t))^k=\sum_{n=k}^{\infty}S_1(n,k)\frac{t^n}{n!},\ (k\geq 0), \text{(see [16-19])}.$$

The degenerate Stirling numbers of the second kind are given by

$$\frac{1}{k!}(e_\lambda(t)-1)^k=\sum_{n=k}^{\infty}S_{2,\lambda}(n,k)\frac{t^n}{n!},\ (k\geq 0), \text{(see [2-24])}. \tag{1.14}$$

It is clear that $\lim_{\lambda\to 0}S_{2,\lambda}(n,k)=S_2(n,k)$, where $S_2(n,k)$ are called the Stirling numbers of the second kind given by

$$\frac{1}{k!}(e^t-1)^k=\sum_{n=k}^{\infty}S_2(n,k)\frac{t^n}{n!},\ (k\geq 0), \text{(see [1-30])}.$$

In this paper, we study the type 2 degenerate multi-poly-Bernoulli polynomials of the second kind arising from modified degenerate multi-polyexponential function, and derive their explicit expressions and some identity involving them. Also, we derive the some identities and relations for the degenerate multi-poly-Bernoulli polynomials of the second kind.

2. Type 2 degenerate multi-poly-Bernoulli polynomials of the second kind

In this section, we define the degenerate multi-poly-Bernoulli polynomials of the second kin and the degenerate multiple polyexponential Bernoulli polynomials of the second kind, respectively as follows.

For $k_1, k_2, \cdots k_r \in \mathbb{Z}$, the degenerate multi-poly-Bernoulli polynomials of the second kind which are degenerate versions of the multi-poly-Bernoulli polynomials in (1.10) and given by

$$\frac{r!\mathrm{Li}_{k_1,k_2,\cdots,k_r}(1-e^{-t})}{[\log_\lambda(1+t)]^r}(1+t)^x = \sum_{n=0}^{\infty} b_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)\frac{t^n}{n!}. \qquad (2.1).$$

When $x=0$, $b_{n,\lambda}^{(k_1,k_2,\cdots k_r)} = b_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(0)$ are called the degenerate multi-poly-Bernoulli numbers of the second kind.

In view of (1.12), and using the degenerate multi polyexponential function, we define the type 2 degenerate multi-poly-Bernoulli polynomials of the second kind by

$$\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}(\log_\lambda(1+t))}{[\log_\lambda(1+t)]^r}(1+t)^x = \sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}, (k \in \mathbb{Z}). \qquad (2.2)$$

When $x=0$, $Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)} = Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(0)$ are called the type 2 degenerate multi-poly-Bernoulli numbers of the second kind.

Note that

$$\lim_{\lambda\to 0}\frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}(\log_\lambda(1+t))}{[\log_\lambda(1+t)]^r}(1+t)^x = \sum_{n=0}^{\infty}\lim_{\lambda\to 0} Pb_{n,\lambda}^{(k)}(k_1,k_2,\cdots,k_r x)\frac{t^n}{n!}$$

$$= \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r}(\log(1+t))}{[\log(1+t)]^r}(1+t)^x = \sum_{n=0}^{\infty} Pb_n^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}, (k \in \mathbb{Z}). \qquad (2.3)$$

By (2.2) and (2.3), we get

$$\lim_{\lambda\to 0} Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = Pb_n^{(k_1,k_2,\cdots,k_r)}(x), (n \geq 0),$$

where $Pb_n^{(k_1,k_2,\cdots,k_r)}(x)$ are called the type 2 multi poly-Bernoulli polynomials of the second kind.

From (2.1), we note that

$$\sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)\frac{t^n}{n!} = \frac{r!\mathrm{Li}_{k_1,k_2,\cdots,k_r}(1-e^{-t})}{[\log_\lambda(1+t)]^r}(1+t)^x$$

$$= \sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}\frac{t^n}{n!}\sum_{m=0}^{\infty}(x)_m\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} Pb_{n-m,\lambda}^{(k_1,k_2,\cdots k_r)}(x)_m \right) \frac{t^n}{n!}. \tag{2.4}$$

Therefore, by equation (2.4), we get the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x) = \sum_{m=0}^{n} \binom{n}{m} Pb_{n-m,\lambda}^{(k_1,k_2,\cdots k_r)}(x)_m.$$

From (2.1), we have

$$\sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)\frac{t^n}{n!} = \frac{r!}{[\log_\lambda(1+t)]^r}\mathrm{Li}_{k_1,k_2,\cdots,k_r}(1-e^{-t})(1+t)^x \tag{2.5}$$

$$= \frac{r!}{[\log_\lambda(1+t)]^r} \sum_{0<n_1<\cdots<n_{r-1}} \frac{1}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}} \sum_{n_r=n_{r-1}+1}^{\infty} \frac{(1-e^{-t})^{n_r}}{n_r^{k_r}}(1+t)^x$$

$$= \frac{r!}{[\log_\lambda(1+t)]^r} \sum_{0<n_1<\cdots<n_{r-1}} \frac{(1-e^{-t})^{n_{r-1}}}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}} \sum_{n_r=1}^{\infty} \frac{(1-e^{-t})^{n_r}}{(n_r+n_{r-1})^{k_r}}(1+t)^x$$

$$= \frac{r!(1+t)^x}{[\log_\lambda(1+t)]^r} \sum_{0<n_1<\cdots<n_{r-1}} \frac{(1-e^{-t})^{n_{r-1}}}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}} \sum_{n_r=1}^{\infty} \frac{(1-e^{-t})^{n_r}}{(n_r+n_{r-1})^{k_r}} \sum_{n=0}^{\infty}(-1)^{l-n_r}S_2(l,n_r)\frac{t^l}{l!}.$$

To proceed further, we observe that for any integer k, we have

$$(x+y)^{-k} = \sum_{m=0}^{\infty}(-1)^m\binom{k+m-1}{m}x^{-k-m}y^m. \tag{2.6}$$

By using equations (2.5) and (2.6), we get

$$\sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)\frac{t^n}{n!} = \frac{rt}{\log_\lambda(1+t)} \left(\frac{(r-1)!(1+t)^x}{[\log_\lambda(1+t)]^{r-1}}\right) \sum_{0<n_1<\cdots<n_{r-1}} \frac{(1-e^{-t})^{n_{r-1}}}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}}$$

$$\times \frac{1}{t}\sum_{l=1}^{\infty}\left(\sum_{n_r=1}^{l} n_r!(-1)^{l-n_r}S_2(l,n_r)\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m n_r^{-k_r-m}\right)\frac{t^l}{l!}$$

$$= \frac{rt}{\log_\lambda(1+t)}\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\sum_{j=0}^{\infty}Pb_{j,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x)\frac{t^j}{j!}$$

$$\times \sum_{l=0}^{\infty}\left(\sum_{n_r=1}^{l+1}\frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)n_r^{-k_r-m}}{l+1}\right)\frac{t^l}{l!}$$

$$= \frac{rt}{\log_\lambda(1+t)}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\binom{k}{l}$$

$$\times \frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)n_r^{-k_r-m}}{l+1}Pb_{k-l,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x)\frac{t^k}{k!}$$

$$= r\sum_{p=0}^{\infty}b_{p,\lambda}\frac{t^p}{p!}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\binom{k}{l}$$

$$\times\frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)n_r^{-k_r-m}}{l+1}Pb_{k-l,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x)\frac{t^k}{k!}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}r\binom{k_r+m-1}{m}(-1)^m\binom{k}{l}\binom{n}{k}$$

$$\times\frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)n_r^{-k_r-m}}{l+1}Pb_{k-l,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x)b_{n-k,\lambda}\frac{t^n}{n!}. \quad (2.7)$$

Therefore, by equation (2.7), we obtain the following theorem.

Theorem 2.2. For $k_1,k_2,\cdots k_r\in\mathbb{Z}$ and $n\geq 0$, we have

$$Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)=r\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\binom{k}{l}\binom{n}{k}$$

$$\times\frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)n_r^{-k_r-m}}{l+1}Pb_{k-l,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x)b_{n-k,\lambda}. \quad (2.8)$$

On replacing k by $-k_r$ in (2.8), we obtain the following corollary.

Corollary 2.1. For $k_1,k_2,\cdots k_r\in\mathbb{Z}$ and $n\geq 0$, we have

$$Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)=r\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{n_r=1}^{l+1}\sum_{m=0}^{\infty}\binom{k_r}{m}(-1)^m\binom{k}{l}\binom{n}{k}$$

$$\times\frac{n_r!(-1)^{l-n_r-1}S_2(l+1,n_r)n_r^{k_r-m}}{l+1}Pb_{k-l,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x)b_{n-k,\lambda}.$$

From (2.1), we have

$$\sum_{n=0}^{\infty}\left(Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x+1)-Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)\right)\frac{t^n}{n!}=\frac{r!}{[\log_\lambda(1+t)]^r}(1+t)^x\mathrm{Li}_{k_1,k_2,\cdots,k_r}(1-e^{-t})$$

$$=\frac{r!t(1+t)^x}{[\log_\lambda(1+t)]^r}\sum_{0<n_1<\cdots<n_{r-1}}\frac{1}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}}\sum_{n_r=n_{r-1}+1}^{\infty}\frac{(1-e^{-t})^{n_r}}{n_r^{k_r}}$$

$$=\frac{r!t(1+t)^x}{[\log_\lambda(1+t)]^r}\sum_{0<n_1<\cdots<n_{r-1}}\frac{(1-e^{-t})^{n_{r-1}}}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}}\sum_{n_r=1}^{\infty}\frac{(1-e^{-t})^{n_r}}{(n_r+n_{r-1})^{k_r}}$$

$$\frac{rt}{\log_\lambda(1+t)}\left(\frac{(r-1)!(1+t)^x}{[\log_\lambda(1+t)]^{r-1}}\right)\sum_{0<n_1<\cdots<n_{r-1}}\frac{(1-e^{-t})^{n_{r-1}}}{n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}}$$

$$\times\sum_{l=1}^{\infty}\left(\sum_{n_r=1}^{l}n_r!(-1)^{l-n_r}S_2(l,n_r)\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m n_r^{-k_r-m}\right)\frac{t^l}{l!}$$

$$=\frac{rt}{\log_\lambda(1+t)}\sum_{m=0}^{\infty}\binom{k_r+m-1}{m}(-1)^m\sum_{j=0}^{\infty}Pb_{j,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x)\frac{t^j}{j!}$$

$$\times\sum_{l=1}^{\infty}\left(\sum_{n_r=1}^{l}n_r!(-1)^{l-n_r}S_2(l,n_r)n_r^{-k_r-m}\right)\frac{t^l}{l!}$$

$$= \frac{rt}{\log_\lambda(1+t)} \sum_{k=0}^{\infty} \sum_{l=1}^{k} \sum_{n_r=1}^{l} \sum_{m=0}^{\infty} \binom{k_r+m-1}{m} (-1)^m \binom{k}{l}$$
$$\times n_r!(-1)^{l-n_r} S_2(l,n_r) n_r^{-k_r-m} Pb_{k-l,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x) \frac{t^k}{k!}$$
$$= r \sum_{p=0}^{\infty} b_{p,\lambda} \frac{t^p}{p!} \sum_{k=0}^{\infty} \sum_{l=1}^{k} \sum_{n_r=1}^{l} \sum_{m=0}^{\infty} \binom{k_r+m-1}{m} (-1)^m \binom{k}{l}$$
$$\times n_r!(-1)^{l-n_r} S_2(l,n_r) n_r^{-k_r-m} Pb_{k-l,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x) \frac{t^k}{k!}$$
$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{l=1}^{k} \sum_{n_r=1}^{l+1} \sum_{m=0}^{\infty} r \binom{k_r+m-1}{m} (-1)^m \binom{k}{l} \binom{n}{k}$$
$$\times n_r!(-1)^{l-n_r} S_2(l,n_r) n_r^{-k_r-m} Pb_{k-l,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x) b_{n-k,\lambda} \frac{t^n}{n!}. \tag{2.9}$$

Therefore, by equation (2.9), we obtain the following theorem.

Theorem 2.3. For $k_1, k_2, \cdots k_r \in \mathbb{Z}$ and $n \geq 0$, we have

$$\frac{1}{r}\left[Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x+1) - Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)\right]$$
$$= \sum_{k=0}^{n} \sum_{l=1}^{k} \sum_{n_r=1}^{l} \sum_{m=0}^{\infty} \binom{k_r+m-1}{m} (-1)^m \binom{k}{l} \binom{n}{k}$$
$$\times n_r!(-1)^{l-n_r} S_2(l,n_r) n_r^{-k_r-m} Pb_{k-l,\lambda}^{(k_1,k_2,\cdots k_{r-1}-m)}(x) b_{n-k,\lambda}. \tag{2.10}$$

Corollary 2.2. For $k_1, k_2, \cdots k_r \in \mathbb{Z}$ and $n \geq 0$, we have

$$Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x+1) - Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x) = nPb_{n-1,\lambda}^{(k_1,k_2,\cdots k_r)}(x).$$

Using (2.1), we have

$$\sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x+y)\frac{t^n}{n!} = \frac{r!\mathrm{Li}_{k_1,k_2,\cdots,k_r}(1-e^{-t})}{[\log_\lambda(1+t)]^r}(1+t)^{x+y}$$
$$= \frac{r!\mathrm{Li}_{k_1,k_2,\cdots,k_r}(1-e^{-t})}{[\log_\lambda(1+t)]^r}(1+t)^x(1+t)^y$$
$$= \sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x)\frac{t^n}{n!} \sum_{n=0}^{\infty} (y)_m \frac{t^m}{m!}$$
$$\sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} Pb_{n-m,\lambda}^{(k_1,k_2,\cdots k_r)}(x)(y)_m \right) \frac{t^n}{n!}. \tag{2.11}$$

Therefore, by equation (2.11), we get the following theorem.

Theorem 2.4. For $k_1, k_2, \cdots k_r \in \mathbb{Z}$ and $n \geq 0$, we have

$$Pb_{n,\lambda}^{(k_1,k_2,\cdots k_r)}(x+y) = \sum_{m=0}^{n} \binom{n}{m} Pb_{n-m,\lambda}^{(k_1,k_2,\cdots k_r)}(x)(y)_m.$$

From (2.2), we have

$$\sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!} = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+t)\right)}{[\log_\lambda(1+t)]^r}(1+t)^x$$

$$= \sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}\frac{t^n}{n!}\sum_{m=0}^{\infty}(x)_m\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}Pb_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)_m\right)\frac{t^n}{n!}. \qquad (2.12)$$

Therefore, by equation (2.12), we get the following theorem.

Theorem 2.5. For $k_1, k_2, \cdots k_r \in \mathbb{Z}$ and $n \geq 0$, we have

$$Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = \sum_{m=0}^{n}\binom{n}{m}Pb_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)_m.$$

From (2.2), we have

$$\sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!} = \frac{r!(1+t)^x}{[\log_\lambda(1+t)]^r}\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}\left(\log_\lambda(1+t)\right)$$

$$= \frac{r!(1+t)^x}{[\log_\lambda(1+t)]^r}\sum_{0<n_1<n_2<\cdots<n_r}\frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}(\log_\lambda(1+t))^{n_r}}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_r^{k_r}}$$

$$= \frac{r!(1+t)^x}{[\log_\lambda(1+t)]^r}\sum_{0<n_1<n_2<\cdots<n_r}\frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_{r-1}}}\frac{(\log_\lambda(1+t))^{n_r}}{n_r!}$$

$$= \frac{r!(1+t)^x}{[\log_\lambda(1+t)]^r}\sum_{0<n_1<n_2<\cdots<n_r\leq m}\frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1.\lambda}(m,n_r)}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_{r-1}}}\frac{t^m}{m!}$$

$$= \frac{1}{t^r}\left(\frac{r!t^r(1+t)^x}{[\log_\lambda(1+t)]^r}\right)\sum_{0<n_1<n_2<\cdots<n_r\leq m}\frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(m,n_r)}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_{r-1}}}\frac{t^m}{m!}$$

$$= \frac{1}{t^r}\sum_{l=0}^{\infty}b_{l,\lambda}^{(r)}(x)\frac{t^l}{l!}\sum_{0<n_1<n_2<\cdots<n_r\leq m}\frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(m,n_r)}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_{r-1}}}\frac{t^m}{m!}$$

$$= \sum_{l=0}^{\infty}b_{l,\lambda}^{(r)}(x)\frac{t^{l-r}}{l!}\sum_{0<n_1<n_2<\cdots<n_r\leq m}\frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(m,n_r)}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_{r-1-1}^{k_{r-1}}n_r^{k_{r-1}}}\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{\binom{n}{l}}{r!\binom{n+r}{n}}b_{l,\lambda}^{(r)}(x)$$

$$\times \sum_{0<n_1<n_2<\cdots<n_r\leq n+r-l}\frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(n+r-l,n_r)}{(n_1-1)!\cdots(n_r-1)!n_1^{k_1}\cdots n_{r-1-1}^{k_{r-1}}n_r^{k_{r-1}}}\frac{t^n}{n!}. \qquad (2.13)$$

Therefore, by equation (2.13), we obtain the following theorem.

Theorem 2.6. For $k_1, k_2, \cdots k_r \in \mathbb{Z}$ and $n \geq 0$, we have

$$Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x) = \sum_{l=0}^{n} \frac{\binom{n+r}{l}}{r!\binom{n+r}{n}} b_{l,\lambda}^{(r)}(x)$$

$$\times \sum_{0<n_1<n_2<\cdots<n_r\leq n+r-l} \frac{(1)_{n_1,\lambda}\cdots(1)_{n_r,\lambda}S_{1,\lambda}(n-l,n_r)}{(n_1-1)!\cdots(n_{r-1}-1)!n_1^{k_1}\cdots n_{r-1}^{k_{r-1}}n_r^{k_{r-1}}}.$$

From (2.2), we have

$$\sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+y)\frac{t^n}{n!} = \frac{r!\mathrm{Ei}_{k_1,k_2,\cdots,k_r,\lambda}(\log_\lambda(1+t))}{[\log_\lambda(1+t)]^r}(1+t)^{x+y}$$

$$= \sum_{n=0}^{\infty} Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty}(y)_m\frac{t^m}{m!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}Pb_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)(y)_m\right)\frac{t^n}{n!}. \tag{2.14}$$

Therefore, by equation (2.14), we obtain the following theorem.

Theorem 2.7. For $k_1, k_2, \cdots k_r \in \mathbb{Z}$ and $n \geq 0$, we have

$$Pb_{n,\lambda}^{(k_1,k_2,\cdots,k_r)}(x+y) = \sum_{m=0}^{n}\binom{n}{m}Pb_{n-m,\lambda}^{(k_1,k_2,\cdots,k_r)}(x)(y)_m.$$

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